

## Invariance of the Tunneling Method for Dynamical Black Holes

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Previous work on dynamical black hole instability is further elucidated within the Hamilton-Jacobi method for horizon tunnelling and the reconstruction of the classical action by means of the null-expansion method and making use of a coordinate invariance approach.

*Keywords:* Dynamical black holes, Hawking radiation.

### 1. Introduction

The Hamilton-Jacobi tunneling method (see, for example<sup>1,2</sup>) is a covariant variant of the Parikh-Wilczek method<sup>3</sup> and it has been extended to the dynamical case.<sup>4-6</sup> The method is based on two natural requirements, namely that the tunnelling rate is an observable and therefore it must be based on invariantly defined quantities, and that coordinate systems which do not cover the horizon should not be admitted. These simple observations can help to clarify some ambiguities, like the doubling of the temperature occurring in the static case when using singular coordinates and the role, if any, of the temporal contribution of the action to the emission rate.

### 2. Hamilton-Jacobi invariant approach to spherically symmetric space-times

Our aim is to study the behavior of a scalar field near a (dynamical) event horizon. The field is governed by the Klein-Gordon equation

$$\square\phi(x^\mu) + m^2 = 0,$$

which, in a WKB approximation, reduces to the relativistic Hamilton-Jacobi equation

$$\phi(x^\mu) = P(x^\mu) \exp\left(-i \frac{I(x^\mu)}{\hbar}\right) \Rightarrow g^{\mu\nu} \partial_\mu I(x^\mu) \partial_\nu I(x^\mu) = 0.$$

Thus, the probability of tunneling through an horizon, if any, to the leading order in  $\hbar$ , is expressed by the relation  $\Gamma \propto \exp - (2/\hbar) \text{Im} I$ . If  $\text{Im} I = \beta\omega$ , with both  $\beta$  and  $\omega$  scalars, and  $\omega$  representing an energy, we can associate a temperature to the field.

If black holes are to be considered real objects, we need to consider them as dynamical (i.e., time-dependent). We take the metric of a spherically symmetric

space-time (SSS) as

$$dS^2 = \gamma_{ij}(x^i)dx^i dx^j + R^2(x^i)d\Omega_2^2.$$

The areal radius  $R(x^i)$  is a geometrically invariant quantity. Other useful quantities are  $\chi(x^i) = \gamma^{ij}\partial_i R \partial_j R$ , which defines the dynamical trapping horizon as  $\chi(x_H) = 0$ , provided  $\partial_i \chi(x_H) \neq 0$ . We can also define the Misner–Sharp mass as  $M = R/2(1 - \chi)$ . Of the utmost importance is the dynamical Hayward’s surface gravity, another scalar quantity:

$$\kappa_H = \frac{1}{2}\square_\gamma R_H.$$

Since we want to perform a WKB approximation, we need to give a definition for the energy of point-like particles, or, equivalently, a family of observers with some properties. Going along previous works, it is natural to choose Kodama observers, since the relation

$$\nabla^b (K^a T_{ab}) = \nabla^b (J_b) = 0$$

ensures the possibility of defining an energy, at least locally, integrating the current  $J^b$  on 3-d spatial hypersurfaces. More in detail, the Kodama vector field can be constructed via

$$K^i = \frac{1}{\sqrt{-\gamma}}\epsilon^{ij}\partial_j R, \quad K^\theta = K^\phi = 0.$$

For a single particle, considering its classical action  $I$ , its invariant energy  $\omega$  will be given by

$$\omega = -K^i \partial_i I.$$

How to reconstruct the action along a selected path in space-time?

$$I = \int_\gamma dx^i \partial_i I,$$

with  $\gamma$  an oriented null curve with at least one point on the dynamical horizon. The integration is split into two parts: one regular outside the horizon, and one that is generally divergent. We need to use a coordinate system regular on the horizon, otherwise the integration has simply no meaning.

According to previous works on the subject, starting from Parikh and Wilczek, we make use of the Feynman regularization for the divergent part of the integral. Thus, we derive

$$\text{Im} I = \left( \text{Im} \int_\gamma dr \frac{\omega}{\kappa_H(r - r_H - i0)} \right) = \frac{\pi \omega_H}{\kappa_H}.$$

$\omega_H$  and  $\kappa_H$  are scalars in the normal space, so the tunneling rate turns out to be an invariant, i.e. an observable, as we could expect.

Is the quantity  $\kappa_H/2\pi$  really a temperature? Of course not, in the strict sense: we are not dealing here with a system in strict thermodynamical equilibrium. However,

we can consider its non-vanishing as a signal of a quantum instability for dynamical black holes regarding quantum emission of particles.

From an experimentalist's point of view, however, under some approximation regarding the slow change of the solution in time, the situation is the same of someone measuring the temperature of a kettle of water which is heated up. Operationally speaking,  $\kappa_H/2\pi$  is the redshift-normalized temperature measured by a Kodama-observer.

In a static setting, one is given at least two choices for defining an invariant energy: the Killing vector is the most discussed one, the Kodama is the other. What normally happens is that

$$\omega_{Killing} = -K_{Killing}^a \partial_a I \neq K_{Kodama}^a \partial_a I = -\omega_{Kodama}.$$

Same happens with the temperature. However, the physical quantity is, strictly speaking,  $\beta\omega$ , and both turn out equal. A thing worth noticing is that Hayward's surface gravity is sensible to conformal transformations, especially to singular ones. This can have an impact on the physics of solutions generated, for example, in different frames of string theory. An example is represented by some stringy solution: in its extremal limit, in the string frame, it's only Hayward's surface gravity which vanishes.

### 3. Concluding remarks

In this contribution we have discussed the so-called Hamilton–Jacobi method through-horizon tunneling. We have dealt with a generic Symmetric Spherically Space-time and made use of three important invariant points, namely regular coordinates on the horizon, a consistent notion of particle energy to be implemented in HJ equations and the null expansions of geodesics for massless particles traveling across the horizon. Using these items, we have shown how the method works in a completely coordinate-invariant way.

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